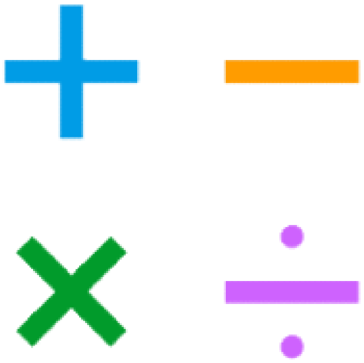
Arithmetic [UNIT-II]



The main arithmetic operations are addition, subtraction, multiplication, and division.

Arithmetic is an elementary branch of mathematics that studies numerical operations like addition, subtraction, multiplication, and division. In a wider sense, it also includes exponentiation, extraction of roots, and taking logarithms.

Arithmetic systems can be distinguished based on the type of numbers they operate on. Integer arithmetic is about calculations with positive and negative integers. Rational number arithmetic involves operations on fractions of integers. Real number arithmetic is about calculations with real numbers, which include both rational and irrational numbers.

Another distinction is based on the numeral system employed to perform calculations. Decimal arithmetic is the most common. It uses the basic numerals from 0 to 9 and their combinations to express numbers. Binary arithmetic, by contrast, is used by most computers and represents numbers as combinations of the basic numerals 0 and 1. Computer arithmetic deals with the specificities of the implementation of binary arithmetic on computers. Some arithmetic systems operate on mathematical objects other than numbers, such as interval arithmetic and matrix arithmetic.

Arithmetic operations form the basis of many branches of mathematics, such as algebra, calculus, and statistics. They play a similar role in the sciences, like physics and economics. Arithmetic is present in many aspects of daily life, for example, to calculate change while shopping or to manage personal finances. It is one of the earliest forms of mathematics education that students encounter. Its cognitive and conceptual foundations are studied by psychology and philosophy.

The practice of arithmetic is at least thousands and possibly tens of thousands of years old. Ancient civilizations like the Egyptians and the Sumerians invented numeral systems to solve practical arithmetic problems in about 3000 BCE. Starting in the 7th and 6th centuries BCE, the ancient Greeks initiated a more abstract study of numbers and introduced the method of rigorous mathematical proofs. The ancient Indians developed the concept of zero and the decimal system, which Arab mathematicians further refined and spread to the Western world during the medieval period. The first mechanical calculators were invented in the 17th century. The 18th and 19th centuries saw the development of modern number theory and the formulation of axiomatic foundations of arithmetic. In the 20th century, the emergence of electronic calculators and computers revolutionized the accuracy and speed with which arithmetic calculations could be performed.

Definition, etymology, and related fields

Arithmetic is the fundamental branch of mathematics that studies numbers and their operations. In particular, it deals with numerical calculations using the arithmetic operations of addition, subtraction, multiplication, and division.[1] In a wider sense, it also includes exponentiation, extraction of roots, and logarithm.[2] The term "arithmetic" has its root in the Latin term "*arithmetica*" which derives from the Ancient Greek words ἀριθμός (arithmos), meaning "number", and ἀριθμητική τέχνη (arithmetike tekhne), meaning "the art of counting".[3]

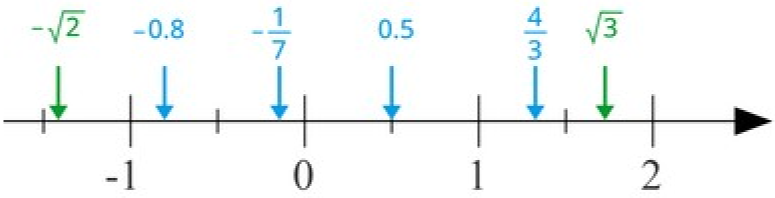
There are disagreements about its precise definition. According to a narrow characterization, arithmetic deals only with natural numbers.[4] However, the more common view is to include operations on integers, rational numbers, real numbers, and sometimes also complex numbers in its scope.[5] Some definitions restrict arithmetic to the field of numerical calculations.[6] When understood in a wider sense, it also includes the study of how the concept of numbers developed, the analysis of properties of and relations between numbers, and the examination of the axiomatic structure of arithmetic operations.[7]

Arithmetic is closely related to number theory and some authors use the terms as synonyms.[8] However, in a more specific sense, number theory is restricted to the study of integers and focuses on their properties and relationships such as divisibility, factorization, and primality.[9] Traditionally, it is known as higher arithmetic.[10]

Numbers

Numbers are mathematical objects used to count quantities and measure magnitudes. They are fundamental elements in arithmetic since all arithmetic operations are performed on numbers. There are different kinds of numbers and different numeral systems to represent them.[11]

Kinds

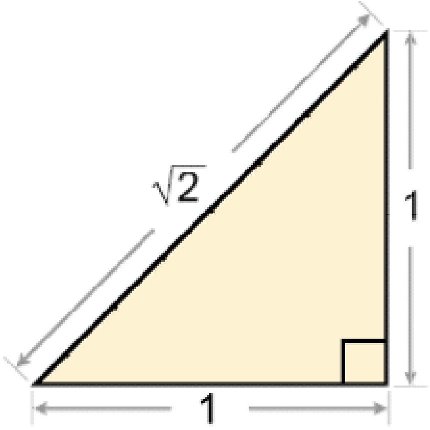


Different types of numbers on a number line. Integers are black, rational numbers are blue, and irrational numbers are green.

The main kinds of numbers employed in arithmetic are natural numbers, whole numbers, integers, rational numbers, and real numbers.[12] The natural numbers are whole numbers that start from 1 and go to infinity. They exclude 0 and negative numbers. They are also known as counting numbers and can be expressed as . The symbol of the natural numbers is .[a] The whole numbers are identical to the natural numbers with the only difference being that they include 0. They can be represented as and have the symbol .[14][b] Some mathematicians do not draw the distinction between the natural and the whole numbers by including 0 in the set of natural numbers.[16] The set of integers encompasses both positive and negative whole numbers. It has the symbol and can be expressed as .[17]

Based on how natural and whole numbers are used, they can be distinguished into cardinal and ordinal numbers. Cardinal numbers, like one, two, and three, are numbers that express the quantity of objects. They answer the question "how many?". Ordinal numbers, such as first, second, and third, indicate order or placement in a series. They answer the question "what position?".[18]

A number is rational if it can be represented as the ratio of two integers. For instance, the rational number is formed by dividing the integer 1, called the numerator, by the integer 2, called the denominator. Other examples are and . The set of rational numbers includes all integers, which are fractions with a denominator of 1. The symbol of the rational numbers is .[19] Decimal fractions like 0.3 and 25.12 are a special type of rational numbers since their denominator is a power of 10. For instance, 0.3 is equal to , and 25.12 is equal to .[20] Every rational number corresponds to a finite or a repeating decimal.[21][c]

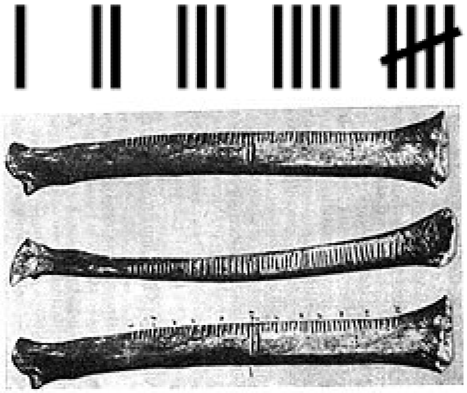


Irrational numbers are sometimes required to describe magnitudes in geometry. For example, the length of the hypotenuse of a right triangle is irrational if its legs have a length of 1.

Irrational numbers are numbers that cannot be expressed through the ratio of two integers. They are often required to describe geometric magnitudes. For example, if a right triangle has legs of the length 1 then the length of its hypotenuse is given by the irrational number . π is another irrational number and describes the ratio of a circle's circumference to its diameter.[22] The decimal representation of an irrational number is infinite without repeating decimals.[23] The set of rational numbers together with the set of irrational numbers makes up the set of real numbers. The symbol of the real numbers is .[24] Even wider classes of numbers include complex numbers and quaternions.[25]

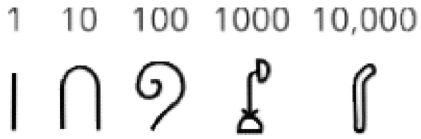
Numeral systems

A numeral is a symbol to represent a number and numeral systems are representational frameworks.[26] They usually have a limited amount of basic numerals, which directly refer to certain numbers. The system governs how these basic numerals may be combined to express any number.[27] Numeral systems are either positional or non-positional. All early numeral systems were non-positional.[28] For non-positional numeral systems, the value of a digit does not depend on its position in the numeral.[29]



Tally marks and some tally sticks use the non-positional unary numeral system.

The simplest non-positional system is the unary numeral system. It relies on one symbol for the number 1. All higher numbers are written by repeating this symbol. For example, the number 7 can be represented by repeating the symbol for 1 seven times. This system makes it cumbersome to write large numbers, which is why many non-positional systems include additional symbols to directly represent larger numbers.[30] Variations of the unary numeral systems are employed in tally sticks using dents and in tally marks.[31]



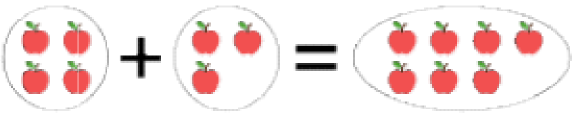
Hieroglyphic numerals from 1 to 10,000[32]

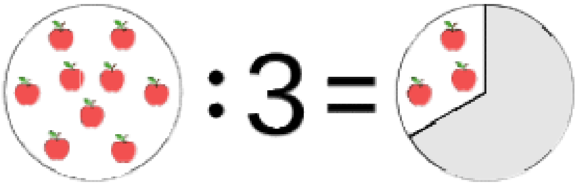
Egyptian hieroglyphics had a more complex non-positional numeral system. They have additional symbols for numbers like 10, 100, 1000, and 10,000. These symbols can be combined into a sum to more conveniently express larger numbers. For instance, the numeral for 10,405 uses one time the symbol for 10,000, four times the symbol for 100, and five times the symbol for 1. A similar well-known framework is the Roman numeral system. It has the symbols I, V, X, L, C, D, M as its basic numerals to represent the numbers 1, 5, 10, 50, 100, 500, and 1000.[33]

A numeral system is positional if the position of a basic numeral in a compound expression determines its value. Positional numeral systems have a radix that acts as a multiplicand of the different positions. For each subsequent position, the radix is raised to a higher power. In the common decimal system, also called the Hindu–Arabic numeral system, the radix is 10. This means that the first digit is multiplied by , the next digit is multiplied by , and so on. For example, the decimal numeral 532 stands for . Because of the effect of the digits' positions, the numeral 532 differs from the numerals 325 and 253 even though they have the same digits.[34]

Another positional numeral system used extensively in computer arithmetic is the binary system, which has a radix of 2. This means that the first digit is multiplied by , the next digit by , and so on. For example, the number 13 is written as 1101 in the binary notation, which stands for . In computing, each digit in the binary notation corresponds to one bit.[35] The earliest positional system was developed by ancient Babylonians and had a radix of 60.[36]

Operations





Arithmetic operations underly many everyday occurrences, like when putting four apples from one bag together with three apples from another bag (top image) or when distributing nine apples equally among three children (bottom image).

Arithmetic operations are ways of combining, transforming, or manipulating numbers. They are functions that have numbers both as input and output.[37] The most important operations in arithmetic are addition, subtraction, multiplication, and division.[38] Further operations include exponentiation, extraction of roots, and logarithm.[39] If these operations are performed on variables rather than numbers, they are sometimes referred to as algebraic operations.[40]

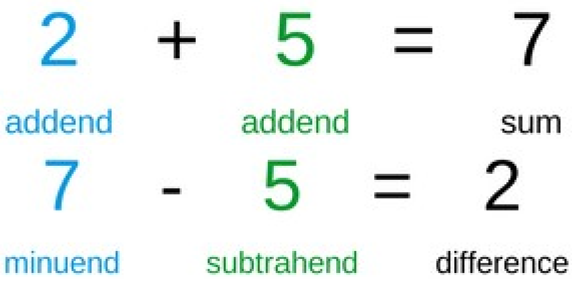
Two important concepts in relation to arithmetic operations are identity elements and inverse elements. The identity element or neutral element of an operation does not cause any change if it is applied to another element. For example, the identity element of addition is 0 since any sum of a number and 0 results in the same number. The inverse element is the element that results in the identity element when combined with another element. For instance, the additive inverse of the number 6 is -6 since their sum is 0.[41]

There are not only inverse elements but also inverse operations. In an informal sense, one operation is the inverse of another operation if it undoes the first operation. For example, subtraction is the inverse of addition since a number returns to its original value if a second number is first added and subsequently subtracted, as in .

Defined more formally, the operation " " is an inverse of the operation " " if it fulfills the following condition: if and only if .[42]

Commutativity and associativity are laws governing the order in which some arithmetic operations can be carried out. An operation is commutative if the order of the arguments can be changed without affecting the results. This is the case for addition, for instance, is the same as . Associativity is a rule that affects the order in which a series of operations can be carried out. An operation is associative if, in a series of two operations, it does not matter which operation is carried out first. This is the case for multiplication, for example, since is the same as .[43]

Addition and subtraction



Addition and subtraction

Addition is an arithmetic operation in which two numbers, called the addends, are combined into a single number, called the sum. The symbol of addition is . Examples are and .[44] The term summation is used if several additions are performed in a row.[45] Counting is a type of repeated addition in which the number 1 is continuously added.[46]

Subtraction is the inverse of addition. In it, one number, known as the subtrahend, is taken away from another, known as the minuend. The result of this operation is called the difference. The symbol of subtraction is .[47] Examples are and . Subtraction is often treated as a special case of addition: instead of subtracting a positive number, it is also possible to add a negative number. For instance . This helps to simplify mathematical computations by reducing the number of basic arithmetic operations needed to perform calculations.[48]

The additive identity element is 0 and the additive inverse of a number is the negative of that number. For instance, and . Addition is both commutative and associative.[49]

Multiplication and division

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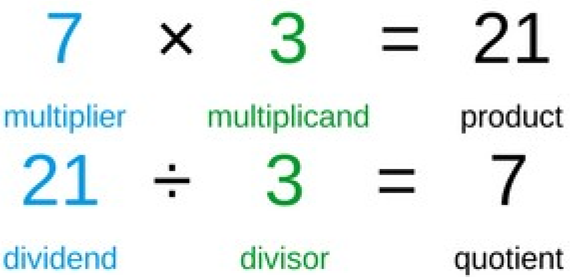


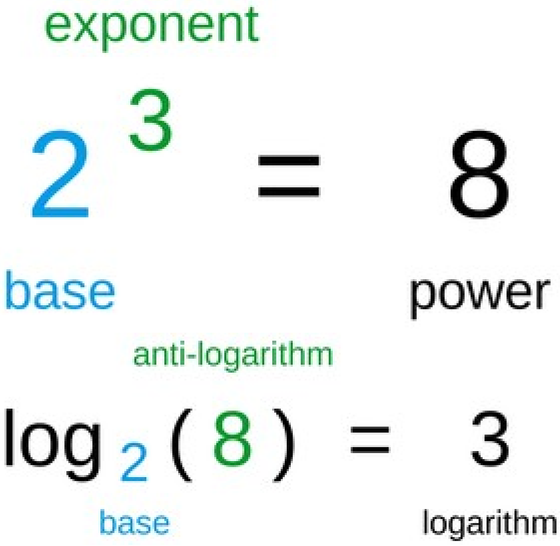
Multiplication and division

Multiplication is an arithmetic operation in which two numbers, called the multiplier and the multiplicand, are combined into a single number called the product.[50][d] The symbols of multiplication are , , and *. Examples are and . If the multiplicand is a natural number then multiplication is the same as repeated addition, as in .[52]

Division is the inverse of multiplication. In it, one number, known as the dividend, is split into several equal parts by another number, known as the divisor. The result of this operation is called the quotient. The symbols of division are and . Examples are and .[53] Division is often treated as a special case of multiplication: instead of dividing by a number, it is also possible to multiply by its reciprocal. The reciprocal of a number is 1 divided by that number. For instance, .[54]

The multiplicative identity element is 1 and the multiplicative inverse of a number is the reciprocal of that number. For example, and . Multiplication is both commutative and associative.[55]

Exponentiation and logarithm



Exponentiation and logarithm

Exponentiation is an arithmetic operation in which a number, known as the base, is raised to the power of another number, known as the exponent. The result of this operation is called the power. Exponentiation is sometimes expressed using the symbol ^ but the more common way is to write the exponent in superscript right after the base. Examples are and ^ . If the exponent is a natural number then exponentiation is the same as repeated multiplication, as in .[56][e]

Roots are a special type of exponentiation using a fractional exponent. For example, the square root of a number is the same as raising the number to the power of and the cube root of a number is the same as raising the number to the power of . Examples are and .[58]

Logarithm is the inverse of exponentiation. The logarithm of a number to the base is the exponent to which must be raised to produce . For instance, since , the logarithm base 10 of 1000 is 3. The logarithm of to base is denoted as , or without parentheses, , or even without the explicit base, , when the base can be understood from context. So, the previous example can be written .[59]

Exponentiation and logarithm do not have general identity elements and inverse elements like addition and multiplication. The neutral element of exponentiation in relation to the exponent is 1, as in . However, exponentiation does not have a general identity element since 1 is not the neutral element for the base.[60] Exponentiation and logarithm are neither commutative nor associative.[61]

Types

Different types of arithmetic systems are discussed in the academic literature. They differ from each other based on what type of number they operate on, what numeral system they use to represent them, and whether they operate on mathematical objects other than numbers.[62]

Integer arithmetic

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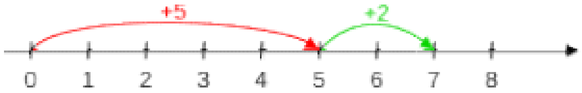


Using the number line method, calculating is performed by starting at the origin of the number line then moving five units to right for the first addend. The result is reached by moving another two units to the right for the second addend.

Integer arithmetic is the branch of arithmetic that deals with the manipulation of positive and negative whole numbers.[63] Simple one-digit operations can be performed by following or memorizing a table that presents the results of all possible combinations, like an addition table or a multiplication table. Other common methods are verbal counting and finger-counting.[64]

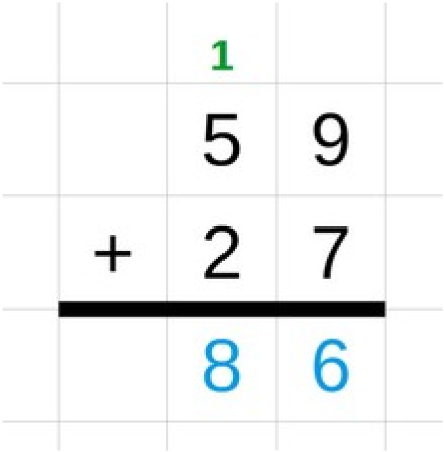
Addition table

+	**0**	**1**	**2**	**3**	**4**	**...**
0	0	1	2	3	4	...
1	1	2	3	4	5	...
2	2	3	4	5	6	...
3	3	4	5	6	7	...
4	4	5	6	7	8	...
...	...	...	...	...	...	...

Multiplication table

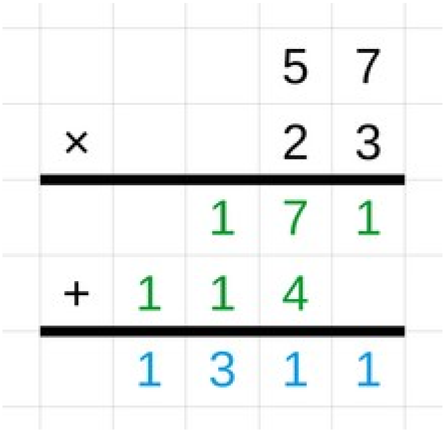
×	**0**	**1**	**2**	**3**	**4**	**...**
0	0	0	0	0	0	...
1	0	1	2	3	4	...
2	0	2	4	6	8	...

3	0	3	6	9	12	...
4	0	4	8	12	16	...
...	...	...	...	...	...	...



Example of addition with carry. The black numbers are the addends, the green number is the carry, and the blue number is the sum.

For operations on numbers with more than one digit, different techniques can be employed to calculate the result by using several one-digit operations in a row. For example, in the method addition with carries, the two numbers are written one above the other. Starting from the rightmost digit, each pair of digits is added together. The rightmost digit of the sum is written below them. If the sum is a two-digit number then the leftmost digit, called the "carry", is added to the next pair of digits to the left. This process is repeated until all digits have been added.[65] Other methods used for integer additions are the number line method, the partial sum method, and the compensation method.[66] A similar technique is utilized for subtraction: it also starts with the rightmost digit and uses a "borrow" or a negative carry for the column on the left if the result of the one-digit subtraction is negative.[67]



			5	7
×			2	3
		1	7	1
+	1	1	4	
	1	3	1	1

Example of long multiplication. The black numbers are the multiplier and the multiplicand. The green numbers are intermediary products gained by multiplying the multiplier with only one digit of the multiplicand. The blue number is the total product calculated by adding the intermediary products.

A basic technique of integer multiplication employs repeated addition. For example, the product of can be calculated as .[68] A common technique for multiplication with larger numbers is called long multiplication. This method starts by writing the multiplier above the multiplicand. The calculation begins by multiplying the multiplier only with the rightmost digit of the multiplicand and writing the result below, starting in the rightmost column. The same is done for each digit of the multiplicand and the result in each case is shifted one position to the left. As a final step, all the individual products are added to arrive at the total product of the two multi-digit numbers.[69] Other techniques used for multiplication are the grid method and the lattice method.[70] Computer science is interested in multiplication algorithms with a low computational complexity to be able to efficiently multiply very large integers, such as the Karatsuba algorithm, the Schönhage–Strassen algorithm, and the Toom–Cook algorithm.[71] A common technique used for division is called long division. Other methods include short division and chunking.[72]

Integer arithmetic is not closed under division. This means that when dividing one integer by another integer, the result is not always an integer. For instance, 7 divided by 2 is not a whole number but 3.5.[73] One way to ensure that the result is an integer is to round the result to a whole number. However, this method leads to inaccuracies as the original value is altered.[74] Another method is to perform the division only partially and retain the remainder. For example, 7 divided by 2 is 3 with a remainder of 1. These difficulties are avoided by rational number arithmetic, which allows for the exact representation of fractions.[75]

A simple method to calculate exponentiation is by repeated multiplication. For instance, the exponentiation of can be calculated as .[76] A more efficient technique used for large exponents is exponentiation by squaring. It breaks down the calculation into a number of squaring operations. For example, the exponentiation can be written as . By taking advantage of repeated squaring operations, only 7 individual operations are needed rather than the 64 operations required for regular repeated multiplication.[77] Methods to calculate logarithms include the Taylor series and continued fractions.[78] Integer arithmetic is not closed under logarithm and under exponentiation with negative exponents, meaning that the result of these operations is not always an integer.[79]

Number theory

Number theory studies the structure and properties of integers as well as the relations and laws between them.[80] Some of the main branches of modern number theory include elementary number theory, analytic number theory, algebraic number theory, and geometric number theory.[81] Elementary number theory studies aspects of integers that can be investigated using elementary methods. Its topics include divisibility, factorization, and primality.[82] Analytic number theory, by contrast, relies on techniques from analysis and calculus. It examines problems like how prime numbers are distributed and the claim that every even number is a sum of two prime numbers.[83] Algebraic number theory employs algebraic structures to analyze the properties of and relations between numbers. Examples are the use of fields and rings, as in algebraic number fields like the ring of integers. Geometric number theory uses concepts from geometry to study numbers. For instance, it investigates how lattice points with integer coordinates behave in a plane.[84] Further branches of number theory are probabilistic number theory, which employs methods from probability theory,[85] combinatorial number theory, which relies on the field of combinatorics,[86] computational number theory, which approaches number-theoretic problems with computational methods,[87] and applied number theory, which examines the application of number theory to fields like physics, biology, and cryptography.[88]

Influential theorems in number theory include the fundamental theorem of arithmetic, Euclid's theorem, and Fermat's last theorem.[89] According to the fundamental theorem of arithmetic, every integer greater than 1 is either a prime number or can be represented as a unique product of prime numbers. For example, the number 18 is not a prime number and can be represented as , all of which are prime numbers. The number 19, by contrast, is a prime number that has no other prime factorization.[90] Euclid's theorem states that there are infinitely many prime numbers.[91] Fermat's last theorem is the statement that no positive integer values can be found for , , and , to solve the equation if is greater than .[92]

Rational number arithmetic

Rational number arithmetic is the branch of arithmetic that deals with the manipulation of numbers that can be expressed as a ratio of two integers.[93] Most arithmetic operations on rational numbers can be calculated by performing a series of integer arithmetic operations on the numerators and the denominators of the involved numbers.

If two rational numbers have the same denominator then they can be added by adding their numerators and keeping the common denominator. For example, . A similar procedure is used for subtraction. If the two numbers do not have the same denominator then they must be transformed to find a common denominator. This can be achieved by scaling the first number with the denominator of the second number while scaling the second number with the denominator of the first number. For instance, .[94]

Two rational numbers are multiplied by multiplying their numerators and their denominators respectively, as in . Dividing one rational number by another can be achieved by multiplying the first number with the reciprocal of the second number. This means that the numerator and the denominator of the second number change position. For example, .[95] Unlike integer arithmetic, rational number arithmetic is closed under division as long as the divisor is not 0.[96]

Both integer arithmetic and rational number arithmetic are not closed under exponentiation and logarithm.[97] One way to calculate exponentiation with a fractional exponent is to perform two separate calculations: one exponentiation using the numerator of the exponent followed by drawing the nth root of the result based on the denominator of the exponent. For example, . The first operation can be completed using methods like repeated multiplication or exponentiation by squaring. One way to get an approximate result for the second operation is to employ Newton's method, which uses a series of steps to gradually refine an initial guess until it reaches the desired level of accuracy.[98] The Taylor series or the continued fraction method can be utilized to calculate logarithms. The decimal fraction notation is a special way of representing rational numbers whose denominator is a power of 10. For instance, the rational numbers , , and are written as 0.1, 3.71, and 0.0044 in the decimal fraction notation.[100] Modified versions of integer calculation methods like addition with carry and long multiplication can be applied to calculations with decimal fractions.[101] Not all rational numbers have a finite representation in the decimal notation. For example, the rational number corresponds to 0.333... with an infinite number of 3s. The shortened notation for this type of repeating decimal is 0.3 Every repeating decimal expresses a rational number

Real number arithmetic

Real number arithmetic is the branch of arithmetic that deals with the manipulation of both rational and irrational numbers. Irrational numbers are numbers that cannot be expressed through fractions or repeated decimals, like the root of 2 and π.[104] Unlike rational number arithmetic, real number arithmetic is closed under exponentiation as long as it uses a positive number as its base. The same is true for the logarithm of positive real numbers as long as the logarithm base is positive and not 1.[105]

Irrational numbers involve an infinite non-repeating series of decimal digits. Because of this, there is often no simple and accurate way to express the results of arithmetic operations like or .[106] In cases where absolute precision is not required, the problem of calculating arithmetic operations on real numbers is usually addressed by truncation or rounding. For truncation, a certain number of leftmost digits are kept and remaining digits are discarded or replaced by zeros. For example, the number π has an infinite number of digits starting with 3.14159.... If this number is truncated to 4 decimal places, the result is 3.141. Rounding is a similar process in which the last preserved digit is increased by one if the next digit is 5 or greater but remains the same if the next digit is less than 5, so that the rounded number is the best approximation of a given precision for the original number. For instance, if the number π is rounded to 4 decimal places, the result is 3.142 because the following digit is a 5, so 3.142 is closer to π than 3.141.[107] These methods allow computers to efficiently perform approximate calculations on real numbers.[108]